Computation of maximal reachability submodules

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Abstract

A new and conceptually simple procedure is derived for the computation of the maximal reachability submodule of a given submodule of the state space of a linear discrete time system over a Noethenian ring R. The procedure is effective if R is effective and if kernels and intersections can be computed. The procedure is compared with a rather different procedure by Assan e.a. published recently.

1 Introduction

Let $A \in R^{n \times n}$, $B \in R^{n \times m}$ where for the moment R is just a commutative ring. As usual, we associate to the pair (A, B) the **linear discrete time control processes**

$$x_0 \quad , \quad x_1 = Ax_0 + Bu_0 \quad , \quad \cdots \quad , \quad x_{k+1} = Ax_k + Bu_k \quad , \quad \cdots \qquad (1)$$

with states $x_k \in R^n$, inputs $u_k \in R^m$ and $k \in \mathbb{N}$.

A submodule U of R^n is called $\boldsymbol{(A, B)}$**-invariant** if $AU \subseteq U + \operatorname{im} B$. An (A, B) invariant submodule U is called **reachable** or **reachability submodule** if every state in U can be reached from zero within U. The latter means:

$$\forall\, x \in U \ \exists\, r \in \mathbb{N},\, u_0, \ldots, u_{r-1} \in R^m : \\ x_1 = Bu_0, \ldots, x_r = A^{r-1}Bu_0 + \ldots + Bu_{r-1} \in U \ \ \textbf{and } x_r = x\,.$$

It was shown (see e.g. [It, Theorem 2.15]) that this rather natural definition is equivalent to the definition of pre-controllability submodules in [CoPe] which is still more commonly known but less intuitive from a control point of view.

The zero-module is trivially (A, B)-invariant and reachable. From the definitions it is clear that sums of (A, B)-invariant or reachable submodules, respectively, are again (A, B)-invariant or reachable. These facts imply that any submodule M of R^n contains a unique maximal (A, B)-invariant submodule M^* and a unique maximal reachability submodule M_0^*, where always $M_0^* \subseteq M^*$.

Maximal reachability submodules play an important role in the solutions to classical control problems such as disturbance decoupling. See [CoPe] and [AsPe] to give only two examples. It is therefore of practical importance to have methods at hand for the computation of generating systems of such modules. In [AsLaPe1] for the first time a finite procedure was given for principal ideal domains and then strongly modified in [AsLaPe2] to work for Noetherian rings. The latter works as follows:

R is now supposed to be Noetherian.

First step (precalculation): $S_0 := \operatorname{im} B$

and for $k \geq 1$: $\quad S_k := \operatorname{im} B + A(S_{k-1} \cap M)$.

This ascending sequence of modules stabilizes after finitely many steps and gives a submodule M_* which contains the image of B. If M is represented as the kernel of some matrix $C \in R^{n \times p}$, then M_* appears as the 'minimal (C, A)-invariant submodule' containing the image of B, see e.g. [AsLaPe2].

Second step and main procedure: $W_0 := M_* \cap M \cap A^{-1}(\operatorname{Im} B)$

and for $k \geq 1$: $\quad W_k := M_* \cap M \cap A^{-1}(W_{k-1} + \operatorname{Im} B)$.

Once more, this gives an ascending sequence and an interesting proof in [AsLaPe2] shows that its limit is actually M_0^*.

Of course - and the same is valid for the new procedure to be developed in this note - such a procedure can be realized in a concrete computation only if the ring R and all the occurring operations like "A^{-1}", "$\cap$" are effective in the sense of [CoCuSt, p.1].

2 New procedure via finite (A, B)-cyclic submodules

Based on results from [BrSch] a quite different and conceptually simpler approach is possible. A submodule U of R^n is called **(A, B)-cyclic** if for some $u_k \in R^m$ and x_k from (1) with $\boldsymbol{x_o = 0}$ one has

$$U = \langle x_k : k \geq 0 \rangle \,. \tag{2}$$

Thus an (A, B)-cyclic submodule can be generated by the states of one single control process which begins with the zero-state.

It is shown in [BrSch] that (A, B)-cyclic submodules are reachability submodules and that **finitely generated** reachability submodules are even **finite** (A, B)-cyclic. The latter means that in addition to (2) one has $x_k = 0$ for $k > d$ and some $d \in \mathbb{N}$.

The point is now that finite (A, B)-cyclic submodules can be determined via the kernel of $[yE - A, -B]$ in $R[y]^{n+m}$. If for $f \in R[y]^n$, $g \in R[y]^m$ one has $(yE - A)f = Bg$, then the coefficient vectors of f generate a finite (A, B)-cyclic submodule and every finite (A, B)-cyclic submodule $U = \langle x_1, \ldots, x_d, 0, \ldots \rangle$ leads to a kernel element $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with $f = x_1 y^{d-1} + \ldots + x_d$ and $g = u_0 y^d + \ldots + u_d$. Note that $x_{d+1} = A_d x_d + B u_d = 0$. More details can be found in [BrSch].

For any $f = x_1 y^{d-1} + \ldots + x_d \in R[y]^n$ let $\boldsymbol{U_f := \langle x_1, \ldots, x_d \rangle}$.
Of course, U_f is contained in a given submodule M if and only if the coefficient vectors of f are from M. Let π be the projection of $R[y]^{n+m} = R[y]^n \oplus R[y]^m$ onto the first n components and let

$$\mathcal{M} := \operatorname{Ker}\big[yE - A, -B\big] \cap \, \big(M[y] \times R[y]^m\big) \,. \tag{3}$$

Here $M[y]$ is the submodule of $R[y]^n$ of those polynomial vectors which have all their coefficient vectors from M.

One arrives now at the following results:

Observation. (i) For every $h \in \mathcal{M}$ the submodule $U_{\pi(h)}$ is a reachability submodule of M (true for any R).

(ii) Let R be Noetherian. For every reachability submodule U of M there is $h \in \mathcal{M}$ such that $U = U_{\pi(h)}$.

Proposition. *Let $h_1, \dots, h_s$ generate $\mathcal{M}$ as an $R[y]$-module, then the family of coefficient vectors of $\pi(h_1), \dots, \pi(h_s)$ generates M_0^*.*

Proof of Observation. (i): By construction $U_{\pi(h)}$ is finite (A, B)-cyclic and thus by Proposition 1.5 in [BrSch] a reachability submodule.

(ii): Since R is Noetherian, U is finitely generated and reachable. By Proposition 1.7 in [BrSch] this implies that U is finite (A, B)-cyclic. The foregoing discussion shows how to construct the desired $h \in \mathcal{M}$. $\square$

Proof of Proposition. Let $f_1 = \pi(h_1), \dots, f_s = \pi(h_s)$ and $\widetilde{M} = \sum_{i=1}^{s} U_{f_i}$. We have to show $\widetilde{M} = M_0^*$. M_0^* is the sum of all reachability submodules of M. Since R is Noetherian, all reachability submodules U of M are finitely generated. By part (ii) of the Observation such modules U can be represented as $U = U_{\pi(h)}$ with some $h \in \mathcal{M}$. Since $h = r_1 h_1 + \ldots + r_s h_s$ with some $r_1, \dots, r_s \in R[y]$, we obtain $U \subseteq \widetilde{M}$ for an arbitrary reachability submodule U of M and thus $M_0^* \subseteq \widetilde{M}$.

The converse inclusion comes from the fact that by part (i) of the Observation U_{f_i} is a reachability submodule of M and therefore contained in M_0^* for $1 \leq i \leq s$. The latter implies: $\widetilde{M} \subseteq M_0^*$. $\square$

One main advantage of the approach via (3) is that one can (for appropriate rings R) compute the kernel of $[yE - A, -B]$ once for all independently of M. This gives us as a first result a module which is of use not only for determining M_0^*, see e.g. [BrSch]. In order to determine M_0^* for some specific M it remains to calculate an intersection of two modules and after that one merely truncates the results and extracts the coefficient vectors.. Explicit calculation is - of course - only possible over an effective Noetherian ring with an effective method to determine the kernel and intersection in (3). Examples of such rings are $\mathbb{Z}, \mathbb{Q}\,[t_1, \dots, t_n]$, $\mathbb{F}\,[t_1, t_n]$ where $\mathbb{F}$ is a finite field. The determination of $\operatorname{Ker}[yE - A, -B]$ can then be done with the help of Gröbner basis calculations as indicated in [BrSch]. A standard technique also via Gröbner bases for the computation of the intersections of modules is (e.g.) described in [AdLou]. In both cases any generating system would do as well. Several current software packages for symbolic computation can be utilized to perform explicit calculations.

A sound comparison of the different procedures for the computation of maximal reachability submodules requires a detailed investigation of their complexities. This remains as a future task.

The following two examples are over $\mathbb{Q}[t]$ and $\mathbb{Q}[t,w]$. Computations have been done combining the well-known packages Macaulay2 and MapleV Release 5.1

Examples

(A) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -t \\ t & t \\ 0 & t \end{bmatrix}$ and $M = \operatorname{im} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ as in Example 1 of [AsLaPe2].

To determine M_0^* we first obtain

$$\operatorname{Ker}[yE - A, -B] = \operatorname{im} \begin{bmatrix} t & -t-y \\ -t & -ty \\ -t & 0 \\ t & -y^2 \\ -y & 0 \end{bmatrix}$$

This leads to $\mathcal{M} = hR[y]$ with $h = {}^t[t,-t,-t,t,-y]$, which in turn leads to with $f = \pi(h) = {}^t[t,-t,-t]$. There is only one coefficient vector to be extracted from f (viewed as a polynomial vector in the variable y). Therefore the final result is: $M_0^* = fR$. By [AsLaPe2] we know $M^* = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} R$ and thus $M_0^* \subsetneqq M^*$.

This example is interesting also since here the classical Wonham-algorithm to determine M^* does not converge and up to now no general finite procedure is known. For principal ideal domains, however, a procedure has been developed in [AsLaLoPe].

(B) In the second example we start with matrices from [AsLaPe2], Example 4.3, where a system with two incommensurable delays is investigated.

Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ w^4 & t & 0 \\ x^3 & t & 1 \end{bmatrix}, \ B = \begin{bmatrix} t & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } M = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & w \\ 0 & 1 \end{bmatrix}.$$

Here Macaulay2 computes

$$\operatorname{Ker}[yE - A, -B] = \operatorname{im} \begin{bmatrix} 0 & -t+y \\ 0 & w^4 \\ -t & -ty+y^2 \\ 1 & 0 \\ -ty & (-w^4t+t^4) - t^3y - ty^2 + y^3 \end{bmatrix},$$

which leads to $\mathcal{M} = hR[y]$ with

$$h = {}^t\left[t^2 - ty,\ -w^4t, -w^3t,\ -w^3 + ty - y^2,\ (w^4t^2 - t^5) + (-w^3t + t^4)y\right].$$

Now $\pi(h) = x_1y + x_2$ where $x_1 = {}^t[-t, 0, 0]$ and ${}^tx_2 = [t^2, -w^4t, -w^3t]$ and according to the Proposition we obtain as final result: $M_0^* = \langle x_1, x_2 \rangle$ (compare with R_2^* in [AsLaPe2, 4.3]). Note that by the new procedure we automatically get M_0^* represented as an (A, B)-cyclic subspace. In more complex examples one obtains M_0^* as a sum of (A, B)-cyclic modules. For reasons of space I do not give an example for this.

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